MAC 2311 Recitation 2 hw prob solutions

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1 Problems

Problem 15 Sec 2.3: Evaluate

$$\lim_{x\to 3}\left[\frac{\sqrt{7-x}-2}{\sqrt{28-x}-5}\right] \tag{1}$$

Solution:

$$\lim_{x\to 3}\left[\frac{\sqrt{7-x}-2}{\sqrt{28-x}-5}\right] = \lim_{x\to 3}\left[\left(\frac{\sqrt{7-x}-2}{\sqrt{28-x}-5}\right)\left(\frac{\sqrt{7-x}+2}{\sqrt{7-x}+2}\right)\right] = \lim_{x\to 3}\left[\frac{7-x-4}{(\sqrt{28-x}-5)(\sqrt{7-x}+2)}\right] \tag{2}$$

$$\Rightarrow \lim_{x\to 3}\left[\frac{3-x}{(\sqrt{28-x}-5)(\sqrt{7-x}+2)}\right] \tag{3}$$

Notice in the denominator, as $x \to 3, \sqrt{28-x}-5 \to 0$

To avoid this,

$$\Rightarrow \lim_{x\to 3}\left[\frac{3-x}{(\sqrt{28-x}-5)(\sqrt{7-x}+2)}\left(\frac{\sqrt{28-x}+5}{\sqrt{28-x}+5}\right)\right] = \lim_{x\to 3}\left[\frac{(3-x)(\sqrt{28-x}+5)}{(\sqrt{7-x}+2)(3-x)}\right] \tag{4}$$

$$\Rightarrow \lim_{x\to 3}\left[\frac{(\sqrt{28-x}+5)}{(\sqrt{7-x}+2)}\right] \Rightarrow \frac{\sqrt{25}+5}{\sqrt{4}+2} = \frac{5+5}{2+2} = \frac{10}{4} = \frac{5}{2} \tag{5}$$

$\square$

Problem 2 Sec 2.4: Use the given graph of $f(x) = \sqrt{x}$ to find a number δ

$$\text{if } |x-4| < \delta \text{ then } |\sqrt{x}-2| < 0.4 \tag{6}$$

Solution:

Definition of continuity:

A function $f : A \to \mathbb{R}$ is continuous at a point $c \in A$ if $\forall \varepsilon > 0, \exists \delta > 0$ such that $\forall x \in A$ where $|x-c| < \delta$ we have that,

$$|f(x) - f(c)| < \varepsilon \tag{7}$$

If f is continuous at every point in its domain, then f is continuous.

Here, we are given that,

$$|x-4| < \delta \tag{8}$$

And,

$$f(x) = \sqrt{x} \tag{9}$$

Furthermore, we are given that,

$$|\sqrt{x} - 2| < 0.4 \tag{10}$$

Thus, let $\varepsilon = 0.4$ so that,

$$|\sqrt{x} - 2| < \varepsilon \tag{11}$$

Work on the LHS of (8). Thus,

$$|x - 4| = |(\sqrt{x} - 2)(\sqrt{x} + 2)| \tag{12}$$

By the triangle inequality,

$$|(\sqrt{x} - 2)(\sqrt{x} + 2)| \leq |\sqrt{x} - 2||\sqrt{x} + 2| \tag{13}$$

Thus, it follows that,

$$|x - 4| \leq |\sqrt{x} - 2||\sqrt{x} + 2| \tag{14}$$

$$\because |\sqrt{x} - 2| < \varepsilon \tag{15}$$

$$\Rightarrow |x - 4| < \varepsilon|\sqrt{x} + 2| \tag{16}$$

Now let us evaluate $|\sqrt{x} + 2|$

Since we had,

$$|\sqrt{x} - 2| < \varepsilon \Rightarrow -\varepsilon < \sqrt{x} - 2 < \varepsilon \tag{17}$$

$$\Rightarrow 2 - \varepsilon < \sqrt{x} < 2 + \varepsilon \tag{18}$$

$$\therefore 4 - \varepsilon < \sqrt{x} + 2 < 4 + \varepsilon \tag{19}$$

We want to utilize the left portion of the inequality.

$$4 - \varepsilon < \sqrt{x} + 2 \tag{20}$$

Then, going back to (16), we have,

$$|x - 4| < \varepsilon(4 - \varepsilon) \tag{21}$$

So, since, $\varepsilon = 0.4$,

$$\Rightarrow |x - 4| < (0.4)(4 - 0.4) \tag{22}$$

$$\therefore |x - 4| < 1.44 \tag{23}$$

$$\because |x - 4| < \delta \tag{24}$$

It follows that $\delta \leq 1.44$

Since by definition $\delta > 0$,

$$\therefore 0 < \delta \leq 1.44 \quad \text{for} \quad 0 \leq \varepsilon \leq 2 \tag{25}$$

□

Alternatively, a quicker proof can be done by utilizing (11). Start by working on the LHS of (11).

$$|\sqrt{x} - 2| = \left|(\sqrt{x} - 2)\left(\frac{\sqrt{x} + 2}{\sqrt{x} + 2}\right)\right| = \frac{|x - 4|}{|\sqrt{x} + 2|} \tag{26}$$

$$\because \frac{|x - 4|}{|\sqrt{x} + 2|} \leq \frac{|x - 4|}{2} \tag{27}$$

$$|\sqrt{x} - 2| \leq \frac{|x - 4|}{2} \tag{28}$$

$$\because |x - 4| < \delta \Rightarrow |\sqrt{x} - 2| < \frac{1}{2}\delta \tag{29}$$

$$\because |\sqrt{x} - 2| < \varepsilon \Rightarrow \varepsilon = \frac{1}{2}\delta \Rightarrow 2\varepsilon = \delta \tag{30}$$

$$\because |\sqrt{x}-2| < \varepsilon \Rightarrow \varepsilon = \frac{1}{2}\delta \Rightarrow 2\varepsilon = \delta \tag{31}$$

$$\because \varepsilon = 0.4 \Rightarrow \delta = 2(0.4) \tag{32}$$

$$\therefore \quad \delta = 0.8 \tag{33}$$

It is worth noting that this answer is still valid because it satisfies the continuity definition. The first proof was done to see what value δ can be at most so that it satisfies the definition. □

Problem 4 Sec 2.6: Find the limit, if it exists. (If an answer does not exist, enter DNE.)

$$\lim_{n\to-\infty}\left[x+\sqrt{x^2+8x}\right] \tag{34}$$

Solution:

$$\lim_{n\to-\infty}\left[x+\sqrt{x^2+8x}\right] = \lim_{n\to-\infty}\left[\left(x+\sqrt{x^2+8x}\right)\left(\frac{x-\sqrt{x^2+8x}}{x-\sqrt{x^2+8x}}\right)\right] \tag{35}$$

$$\Rightarrow \lim_{n\to-\infty}\left[\frac{x^2-x^2-8x}{x-\sqrt{x^2+8x}}\right] = \lim_{n\to-\infty}\left[\frac{-8x}{x-\sqrt{x^2+8x}}\right] = \lim_{n\to-\infty}\left[\frac{-8x}{x-\sqrt{\left(x^2\right)\left(1+\frac{8}{x}\right)}}\right] \tag{36}$$

$$\Rightarrow \lim_{n\to-\infty}\left[\frac{-8x}{x-\sqrt{x^2}\sqrt{\left(1+\frac{8}{x}\right)}}\right] \tag{37}$$

Note that $\sqrt{x^2} = \pm\sqrt{x^2}$ so, $x = |x|$

$$\Rightarrow \lim_{n\to-\infty}\left[\frac{-8x}{x-|x|\sqrt{\left(1+\frac{8}{x}\right)}}\right] \tag{38}$$

$\because |x| = -x$ for $x < 0$

$$\Rightarrow \lim_{n\to-\infty}\left[\frac{-8x}{x-(-x)\sqrt{\left(1+\frac{8}{x}\right)}}\right] = \lim_{n\to-\infty}\left[\frac{-8x}{x+x\sqrt{\left(1+\frac{8}{x}\right)}}\right] = \lim_{n\to-\infty}\left[\frac{-8x}{x\left(1+\sqrt{\left(1+\frac{8}{x}\right)}\right)}\right] \tag{39}$$

$$\therefore \lim_{n\to-\infty}\left[\frac{-8}{\left(1+\sqrt{\left(1+\frac{8}{x}\right)}\right)}\right] = \frac{-8}{2} = -4 \tag{40}$$

□